Fast Polyhedra Abstract Domain



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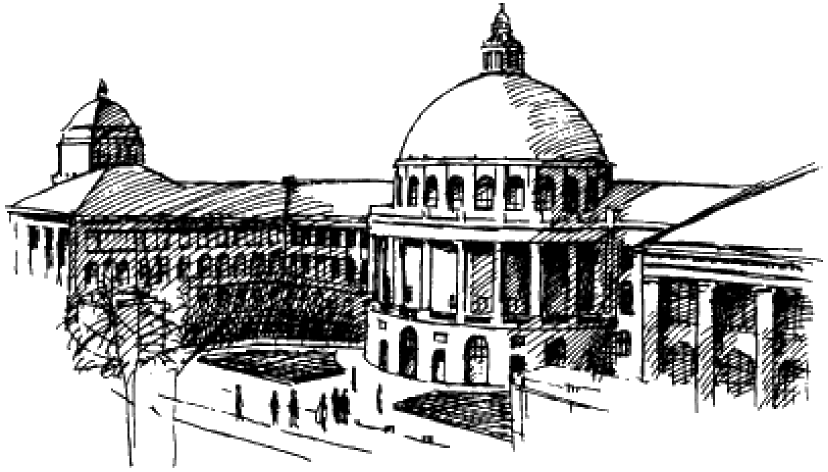

Polyhedra Domain Analysis

Automatic Discovery of Linear Restraints Among Variables of a Program, POPL'78

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Introduced by Patrick Cousot and Nicolas Halbwachs

Represents linear constraints between program variables

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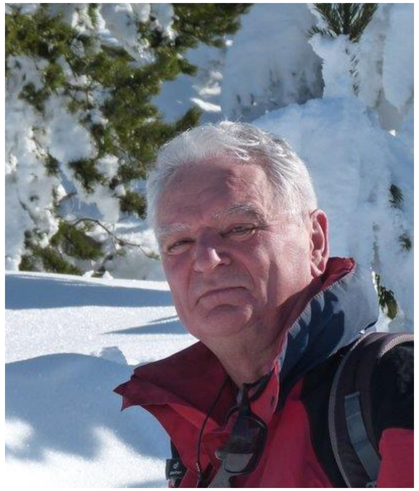

Nicolas Halbwachs

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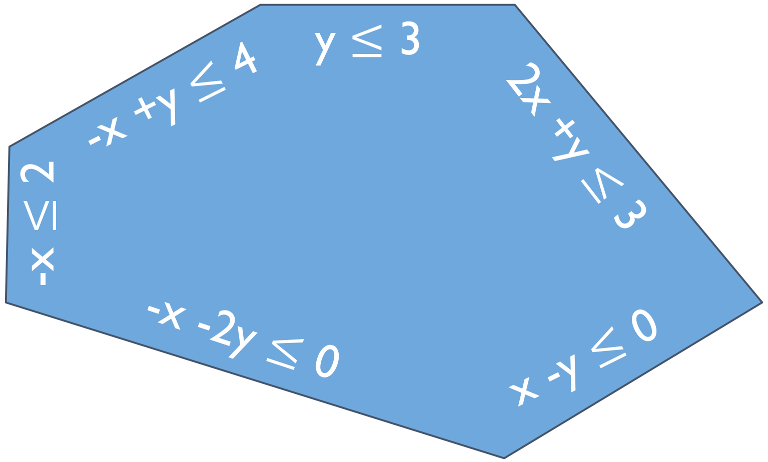

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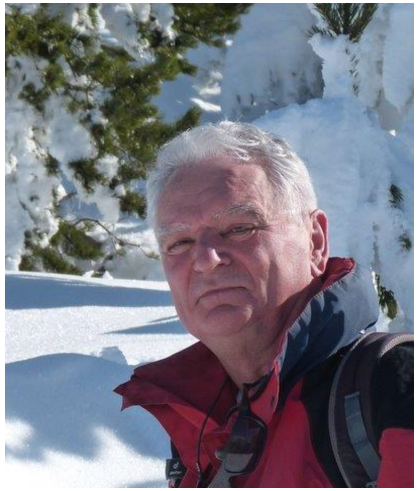

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if(*){
      y:=2x-1;
}
else{
      y:=2x-2;
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assert(y<=2x);
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Abstract Domain	Can Prove the Assertion?
Interval	✗
Pentagon	✗
Zones	✗
Octagon	✗
Polyhedra	✓

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Polyhedra analysis: time and space exponential in number of variables

This work: contributions

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Online decomposition:
reduction in space and time
without losing precision

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Constant factor improvements
via reduced operation count and
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Complete end-to-end implementation

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Constant factor improvements
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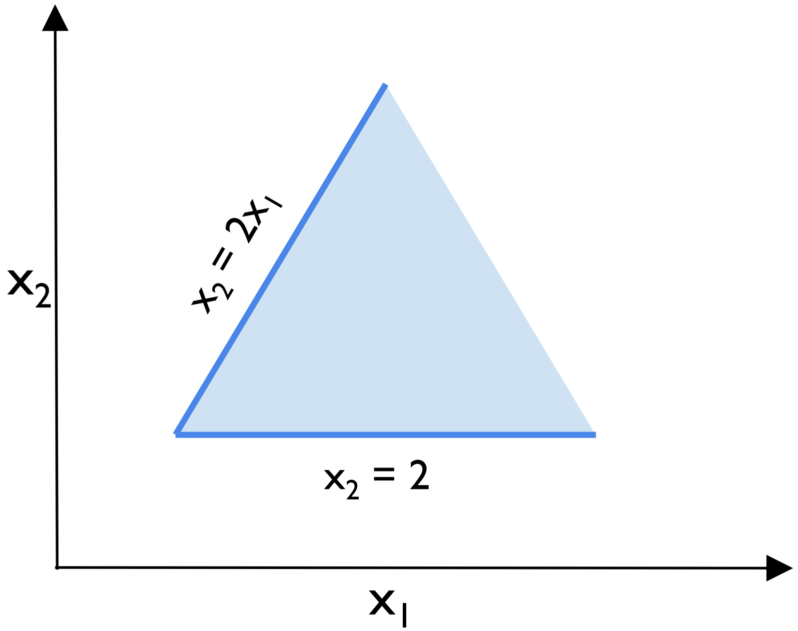
Complete end-to-end implementation

Driver	NewPolka	PPL	ELINA
➢ 500 var ➢ 39K LOC	OOM (> 12 GB)	OOM (> 12 GB)	4 sec 0.9 GB
➢ 650 var ➢ 25K LOC	TO (> 4 hr)	TO (> 4 hr)	2 sec 0.4 GB

Double Representation of Polyhedron

Double Representation of Polyhedron

Constraints

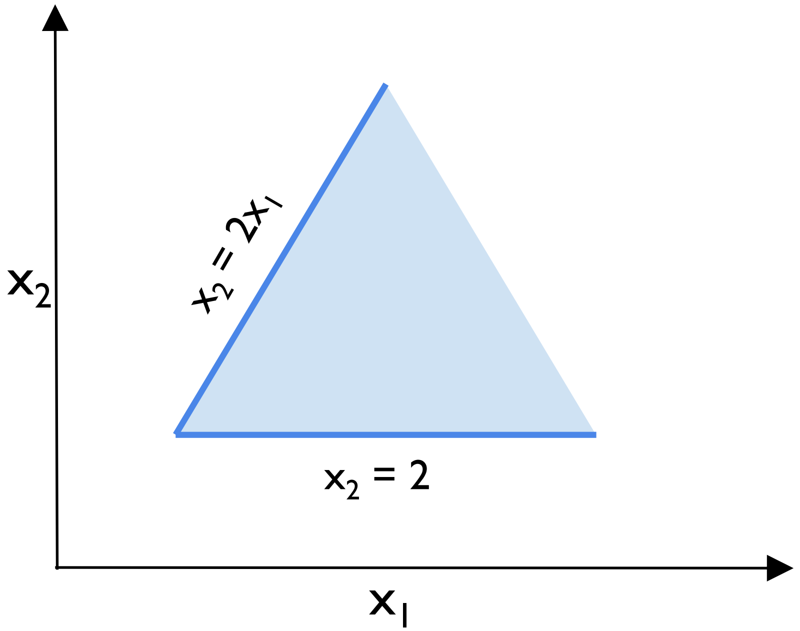


$\mathcal{C}$ = {$-x_2 \leq -2$, $x_2 \leq 2x_1$}

m: number of constraints

Double Representation of Polyhedron

Constraints



$\mathcal{C} = \{-x_2 \leq -2, x_2 \leq 2x_1\}$

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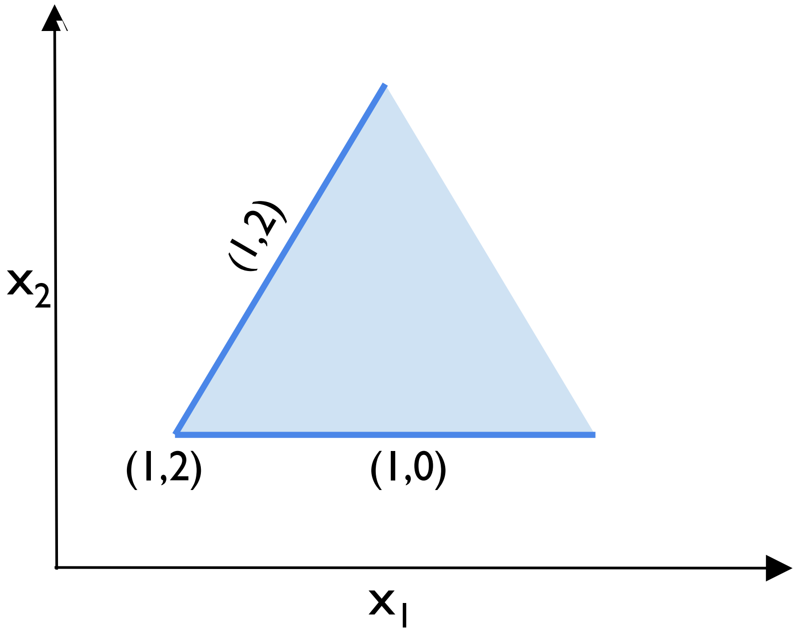
Generators



Vertices $\mathcal{V} = \{(1,2)\}$,

Rays $\mathcal{R} = \{(1,2), (1,0)\}$,

Lines $\mathcal{Z} = \emptyset$

g: number of generators

Asymptotic Time Complexity of Polyhedra

Asymptotic Time Complexity of Polyhedra

Operator	Constraints	Generators	Both
Join (⊔)	exp(n,m)	$O(ng)$	$O(ng)$
Meet (⊓)	$O(nm)$	exp(n,g)	$O(nm)$
Inclusion (⊑)	exp(n,m)	exp(n,g)	$O(ngm)$
Assignment	$O(nm^2)$	$O(ng)$	$O(ng)$
Conditional	$O(n)$	exp(n,g)	$O(n)$

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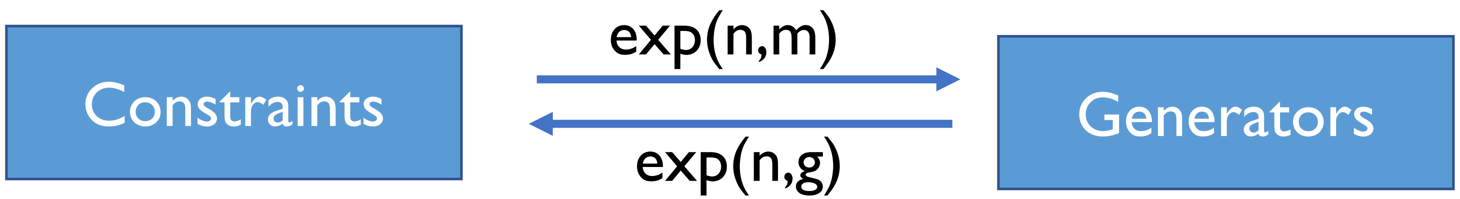

Key Idea: Online Decomposition

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Polyhedron

$\{x_1 \leq 2x_2,$
$x_2 = 2,$
$x_1 + x_2 + 2x_3 \leq 5,$

$x_4 - x_5 \leq 3,$
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Set of factors

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Partition (π) = set of blocks

$\{x_1, x_2, x_3\}$

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working on smaller Polyhedra enables reduction in space and time

Permissible Partitions

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Polyhedron	Best (finest) partition (π)
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$x_4 - x_5 \leq 3,$ $x_5 = 1,$	$\{x_4, x_5\}$	$\{x_4, x_5, x_6\}$	$\{x_3, x_4, x_5\}$
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Definition: *A partition π is permissible for Polyhedron P, if there are no two variables x_i and x_j in different blocks of π related by a constraint in P*

Partition of Variable Set: Summary

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The set of all partitions of variable set $\mathcal{X}$ form a lattice ordered by "*finer than*" (<) relation

The best (finest) partition π_P for Polyhedron P is unique

Any $\overline{\pi}$, s.t., $\pi_P < \overline{\pi}$, is permissible

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Challenge: maintain permissible partitions for > 30 operators

Operator: Conditional

Operator: Conditional

Definition: *Let* π *be a partition and* $\mathcal{B}$ *be a block, then* $\pi \uparrow \mathcal{B}$ *is the finest partition* π' *such that* $\pi \sqsubseteq \pi$' *and* $\mathcal{B}$ *is a subset of an element of* π'

Theorem (finest partition after conditional):

If $O \neq \perp$ *and let* $\mathcal{B}$ *be block containing all variables appearing in the conditional, then* $\pi_O = \pi_P \uparrow \mathcal{B}$

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P	π_P
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if($x_2 \leq 2x_4$) →

O	π_O
$\{x_1 \leq 2x_2,$ $x_2 = 2,$ $x_1 + x_2 + 2x_3 \leq 5,$ $x_4 - x_5 \leq 3,$ $x_5 = 1,$ $x_2 \leq 2x_4\}$	$\{x_1, x_2, x_3,$ $x_4, x_5\}$
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P	π_P		O	π_O
$\{x_1 \le 2x_2,\ x_2 = 2,\ x_1 + x_2 + 2x_3 \le 5\}$	$\{x_1, x_2, x_3\}$	if($x_2 \le 2x_4$) → $\mathcal{B} = \{x_2, x_4\}$	$\{x_1 \le 2x_2,\ x_2 = 2,\ x_1 + x_2 + 2x_3 \le 5,\ x_4 - x_5 \le 3,\ x_5 = 1,\ x_2 \le 2x_4\}$	$\{x_1, x_2, x_3, x_4, x_5\}$
$\{x_4 - x_5 \le 3,\ x_5 = 1\}$	$\{x_4, x_5\}$			
$\{x_6 = 2\}$	$\{x_6\}$		$\{x_6 = 2\}$	$\{x_6\}$

Theorem (finest partition after conditional):

If O $\neq \perp$ and let $\mathcal{B}$ be block containing all variables appearing in the conditional, then $\pi_O = \pi_P \uparrow \mathcal{B}$

Operator: Assignment

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Operator: Assignment

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$x_5 = 1\}$

$\{x_6 = 2\}$

π_P

$\{x_1, x_2, x_3\}$

$\{x_4, x_5\}$

$\{x_6\}$

$x_2 := 2x_4$

Operator: Assignment

P	π_P		O	π_O
$\{x_1 \leq 2x_2,$ $x_2 = 2,$ $x_1 + x_2 + 2x_3 \leq 5\}$	$\{x_1, x_2, x_3\}$		$\{x_1 \leq 4,$ $x_1 + 2x_3 \leq 3,$	$\{x_1, x_3\}$
$\{x_4 - x_5 \leq 3,$ $x_5 = 1\}$	$\{x_4, x_5\}$	$x_2 := 2x_4$ →	$\{x_4 - x_5 \leq 3,$ $x_5 = 1,$ $x_2 = 2x_4\}$	$\{x_2, x_4, x_5\}$
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Theorem (finest partition after assignment):

Let $\mathcal{B}$ be block containing all variables appearing for assignment $\boldsymbol{x_i} := \boldsymbol{e}$, and let π_i = $\{\mathcal{X} \setminus \{\boldsymbol{x_i}\}, \{\boldsymbol{x_i}\}\}$, then $\pi_O = (\pi_P \sqcap \pi_i) \uparrow \mathcal{B}$

Operator: Assignment

P	π_P		O	π_O
$\{x_1 \leq 2x_2,$ $x_2 = 2,$ $x_1 + x_2 + 2x_3 \leq 5\}$	$\{x_1, x_2, x_3\}$		$\{x_1 \leq 4,$ $x_1 + 2x_3 \leq 3,$	$\{x_1, x_3\}$
$\{x_4 - x_5 \leq 3,$ $x_5 = 1\}$	$\{x_4, x_5\}$	$x_2 := 2x_4$ → $\mathcal{B} = \{x_2, x_4\}$	$\{x_4 - x_5 \leq 3,$ $x_5 = 1,$ $x_2 = 2x_4\}$	$\{x_2, x_4, x_5\}$
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Lattice Operators

Lattice Operators

Theorem (finest partition for $\sqsubseteq$):

If $P \sqsubseteq Q$ *and* $P \neq \bot$*, then* $\pi_Q \sqsubseteq \pi_P$

Theorem: **(finest partition after $\sqcap$)**:

If $P \sqcap Q \neq \bot$ *, then* $\pi_O = \pi_P \sqcup \pi_Q$

For join ($\sqcup$), no general relationship exists between π_O, π_P and π_Q

Operator: Join ($\sqcup$)

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P	π_P
$\{x_1 - x_2 \leq 0, x_1 \leq 0\}$	$\{x_1, x_2\}$
$\{x_3 = 1\}$	$\{x_3\}$

Operator: Join ($\sqcup$)

P

$\{x_1 - x_2 \leq 0,$
$x_1 \leq 0\}$

$\{x_3 = 1\}$

π_P

$\{x_1, x_2\}$

$\sqcup$

$\{x_3\}$

Operator: Join ($\sqcup$)

P	π_P		Q	π_Q
$\{x_1 - x_2 \leq 0, x_1 \leq 0\}$	$\{x_1, x_2\}$	$\sqcup$	$\{x_1 \leq 2\}$	$\{x_1\}$
			$\emptyset$	$\{x_2\}$
$\{x_3 = 1\}$	$\{x_3\}$		$\{x_3 = 0\}$	$\{x_3\}$

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P	π_P		Q	π_Q		O	π_O
$\{x_1 - x_2 \leq 0, x_1 \leq 0\}$	$\{x_1, x_2\}$	$\sqcup$	$\{x_1 \leq 2\}$	$\{x_1\}$	$\Rightarrow$	$\{x_1 + 2x_3 \leq 2, -x_3 \leq 0, x_3 \leq 1\}$	$\{x_1, x_3\}$
			$\emptyset$	$\{x_2\}$		$\emptyset$	$\{x_2\}$
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		$\sqcup$	$\emptyset$	$\{x_2\}$	$\Rightarrow$	$\emptyset$	$\{x_2\}$
$\{x_3 = 1\}$	$\{x_3\}$		$\{x_3 = 0\}$	$\{x_3\}$			

$$\pi_P \sqcup \pi_Q = \pi_P \neq \pi_O$$

Operator: Join ($\sqcup$)

P	π_P		Q	π_Q		O	π_O
$\{x_1 - x_2 \leq 0,\ x_1 \leq 0\}$	$\{x_1, x_2\}$	$\sqcup$	$\{x_1 \leq 2\}$	$\{x_1\}$	$\Rightarrow$	$\{x_1 + 2x_3 \leq 2,\ -x_3 \leq 0,\ x_3 \leq 1\}$	$\{x_1, x_3\}$
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P	π_P		Q	π_Q		O	π_O
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$$\pi_P \sqcup \pi_Q = \pi_P \neq \pi_O$$

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For Join, π_O depends on both P and Q

Operator: Join ($\sqcup$)

Operator: Join ($\sqcup$)

Theorem: *Let P and Q be two Polyhedra with the same permissible partition* $\pi = \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_r\}$ *and let* $\overline{\pi}$ *be a permissible partition for the join, that is,* $\pi_{P \sqcup Q} \sqsubseteq \overline{\pi}$. *If for any block* $\mathcal{X}_k \in \pi$, $P_k = Q_k$, *then* $\mathcal{X}_k \in \overline{\pi}$

Operator: Join (⊔)

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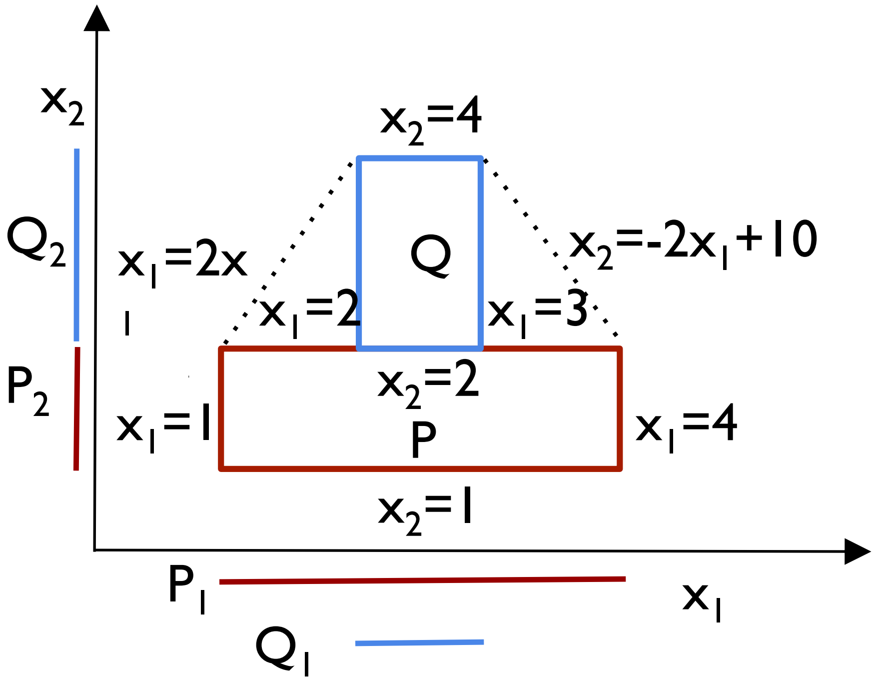
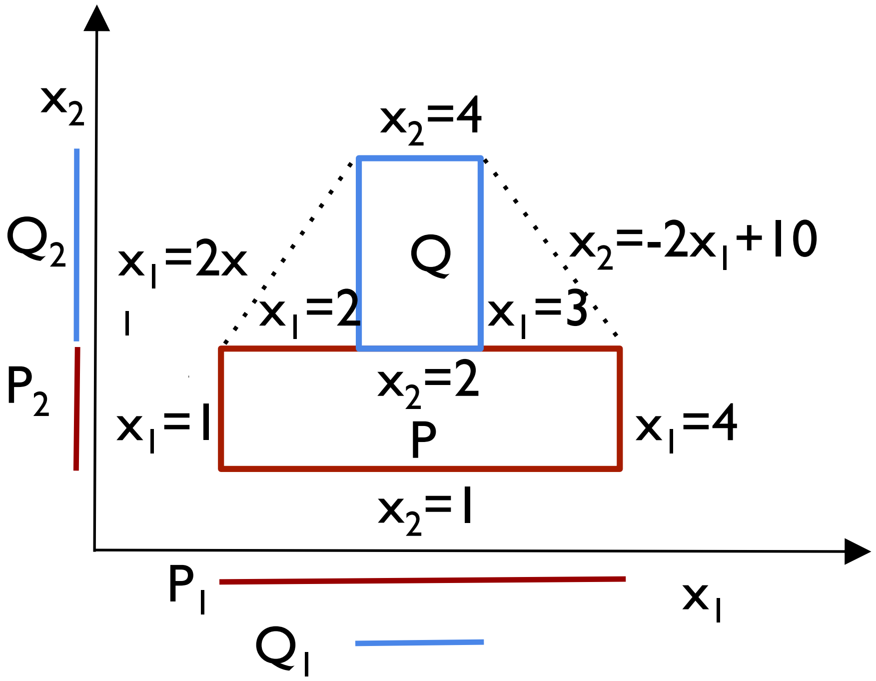
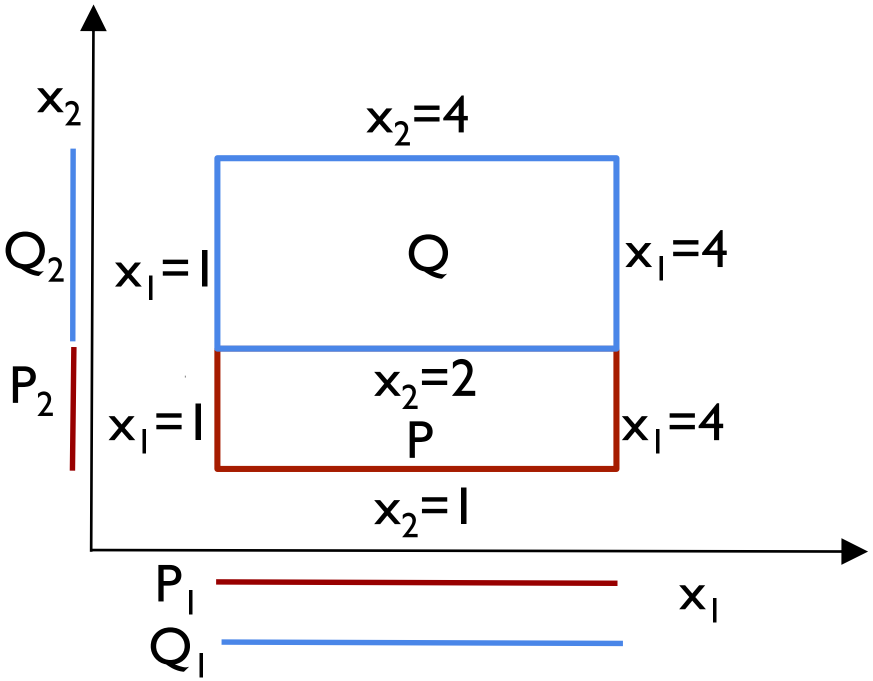


Operator: Join (⊔)

Theorem: *Let P and Q be two Polyhedra with the same permissible partition* $\pi = \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_r\}$ *and let* $\overline{\pi}$ *be a permissible partition for the join, that is,* $\pi_{P \sqcup Q} \sqsubseteq \overline{\pi}$. *If for any block* $\mathcal{X}_k \in \pi$, $P_k = Q_k$, *then* $\mathcal{X}_k \in \overline{\pi}$





Operators with Permissible Partitions

Operators with Permissible Partitions

Theorem (permissible partition after join):

Let $\overline{\pi} = \overline{\pi}_P \sqcup \overline{\pi}_Q$ and $\mathcal{U} = \{\mathcal{X}_k \mid P_k = Q_k, \mathcal{X}_k \in \overline{\pi}\}$.

Then $\overline{\pi}_{P \sqcup Q} = \mathcal{U} \cup \biguplus_{\mathcal{T} \in \overline{\pi} \setminus \mathcal{U}} \mathcal{T}$ *is permissible for* $P \sqcup Q$

Theorem (permissible partition after meet):

$\overline{\pi}_P \sqcup \overline{\pi}_Q$ *is permissible for* $P \sqcap Q$

Theorem (permissible partition after conditional):

If output $O \neq \perp$, then, $\overline{\pi}_P \uparrow \mathcal{B}$ *is permissible for conditional*

Theorem (permissible partition after assignment):

$\overline{\pi}_P \uparrow \mathcal{B}$ *is permissible for the output O of assignment*

Asymptotic Complexity of Operators with Permissible Partitions

Operator	Before (using both)	Our work (using decomposition)
Join ($\sqcup$)	$O(ng)$	$O(\sum_{i=1}^{r} n_i m_i g_i + n_{max} m_{max})$
Meet ($\sqcap$)	$O(nm)$	$O(\sum_{i=1}^{r} n_i m_i)$
Inclusion ($\sqsubseteq$)	$O(ngm)$	$O(\sum_{i=1}^{r} n_i m_i g_i)$
Assignment	$O(ng)$	$O(n_{max} g_{max})$
Conditional	$O(n)$	$O(n_{max})$
Conversion	$exp(n,g)$	$exp(n_{max}, g_{max})$

r: number of blocks

Experimental Evaluation

We compared performance of ELINA against NewPolka and PPL

Using the Seahorn verification framework [CAV'15]

- written in C, analyzes llvm-bitcode
- produces Polyhedra invariants

> 1500 benchmarks from the software verification competition

Time limit: 4 hours

Memory limit: 12 GB

Experimental Evaluation

Experimental Evaluation

Benchmark	Category	LOC	NewPolka		PPL		ELINA		Speedup ELINA vs.	
			time(s)	memory(GB)	time(s)	memory(GB)	time(s)	memory(GB)	NewPolka	PPL
firewire_firedtv	LD	14506	1367	1.7	331	0.9	0.4	0.2	3343	828
net_fddi_skfp	LD	30186	5041	11.2	6142	7.2	9.2	0.9	547	668
mtd_ubi	LD	39334	3633	7	MO	MO	4	0.9	908	>38
usb_core_main0	LD	52152	11084	2.7	4003	1.4	65	2	170	62
tty_synclinkmp	LD	19288	TO	TO	MO	MO	3.4	0.1	>4235	>1186
scsi_advansys	LD	21538	TO	TO	TO	TO	4	0.4	>3600	>3600
staging_vt6656	LD	25340	TO	TO	TO	TO	2	0.4	>7200	>7200
net_ppp	LD	15744	TO	TO	10530	0.15	924	0.3	>16	11.4
p10_l00	CF	592	841	4.2	121	0.9	11	0.8	76	11
p16_l40	CF	1783	MO	MO	MO	MO	11	3	>69	>24
p12_l57	CF	4828	MO	MO	MO	MO	14	0.8	>71	>15
p13_l53	CF	5816	MO	MO	MO	MO	54	2.7	>50	>26
p19_l59	CF	9794	MO	MO	MO	MO	70	1.7	>15	>4
ddv_all	HM	6532	710	1.4	85	0.5	0.05	0.1	12772	1700

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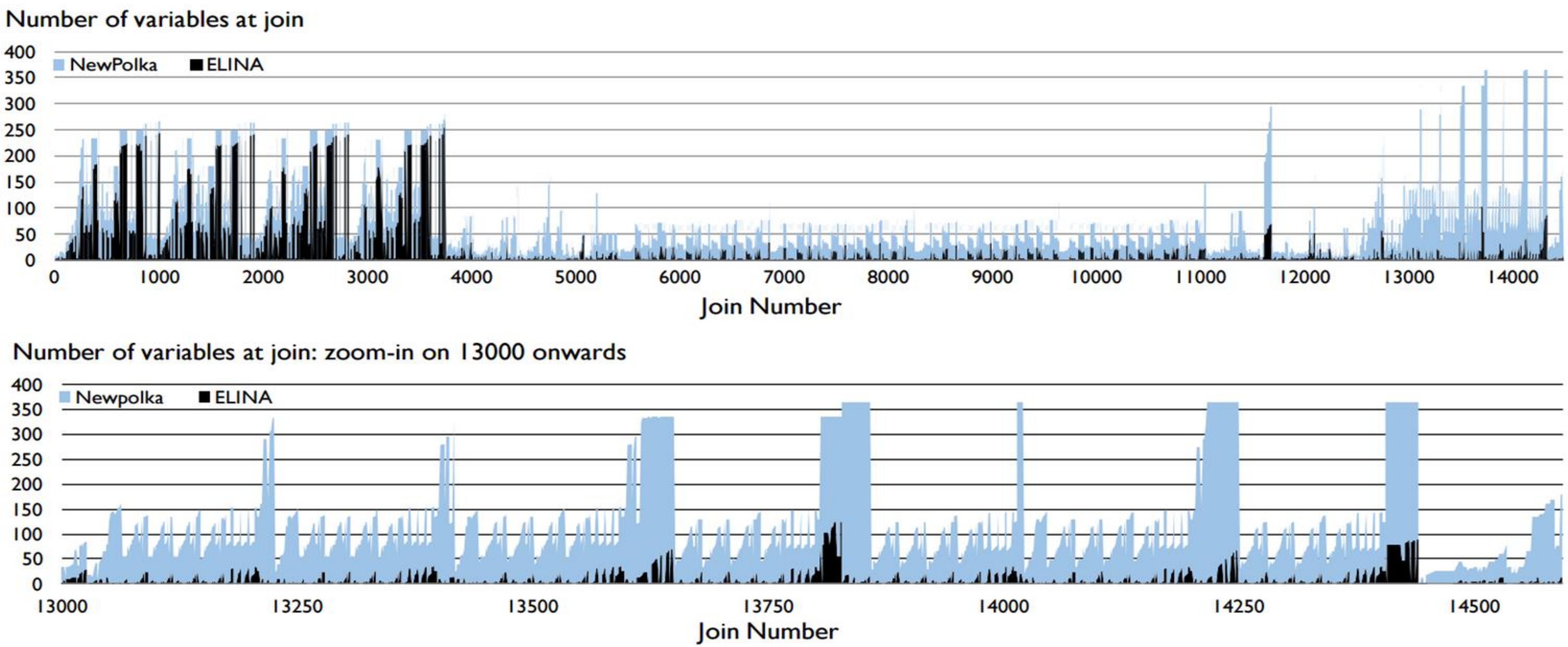
Evaluation



$n_{ELINA} < n_{NewPolka}$, large speedup as conversion is exponential in n

Related Work

Related Work

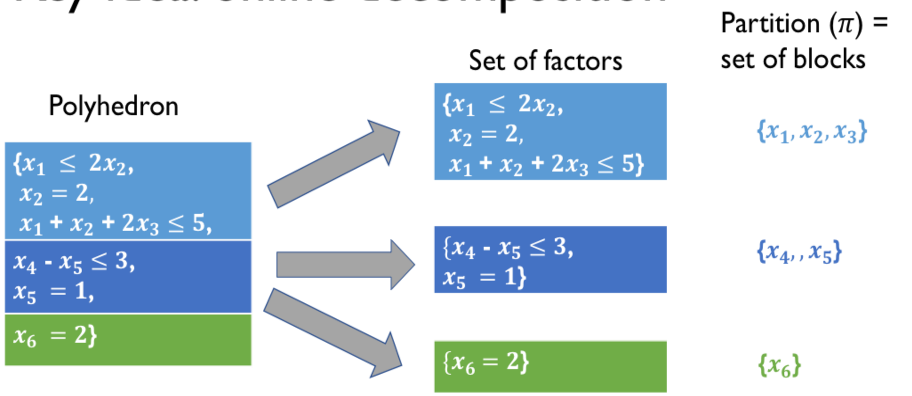
- Variable Packing
  - Blanchet et al. [PLDI’03]
  - decomposition based on syntactic criteria
  - loses precision

- Matrix based decomposition
  - Halbwachs et al. [FMSD’06]
  - does not work with generators
  - decomposition too coarse for join

Conclusion

Conclusion

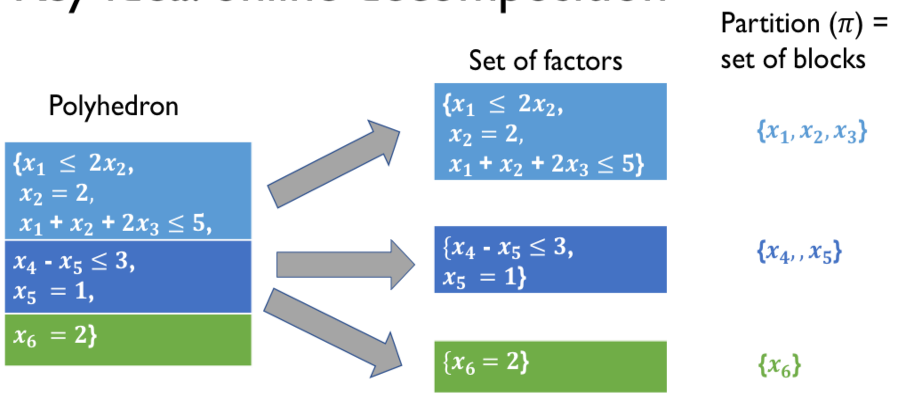
Key idea: online decomposition



Operators work on smaller Polyhedra: Complexity Reduction

Conclusion

Key idea: online decomposition



Operators work on smaller Polyhedra: Complexity Reduction

Operator	Both	Online decomposition
Join ($\sqcup$)	$O(ng)$	$O(\sum_{i=1}^{r} n_i m_i g_i + n_{max} m_{max})$
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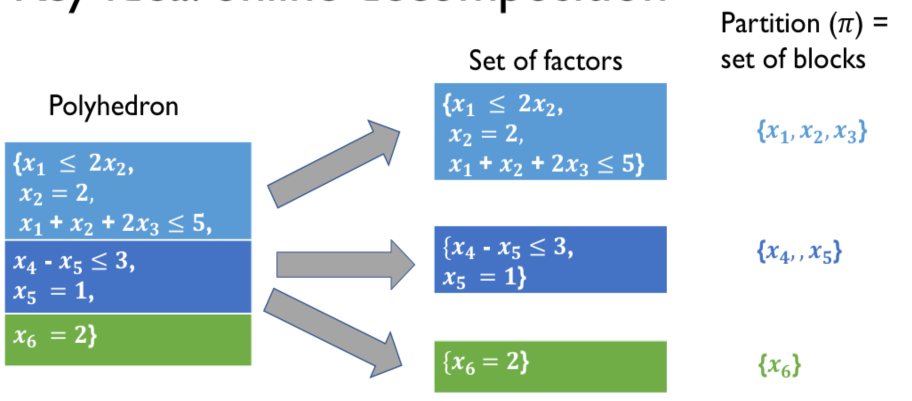
Conclusion

Key idea: online decomposition



Operators work on smaller Polyhedra: Complexity Reduction

Operator	Both	Online decomposition
Join (⊔)	$O(ng)$	$O(\sum_{i=1}^{r} n_i m_i g_i + n_{max} m_{max})$
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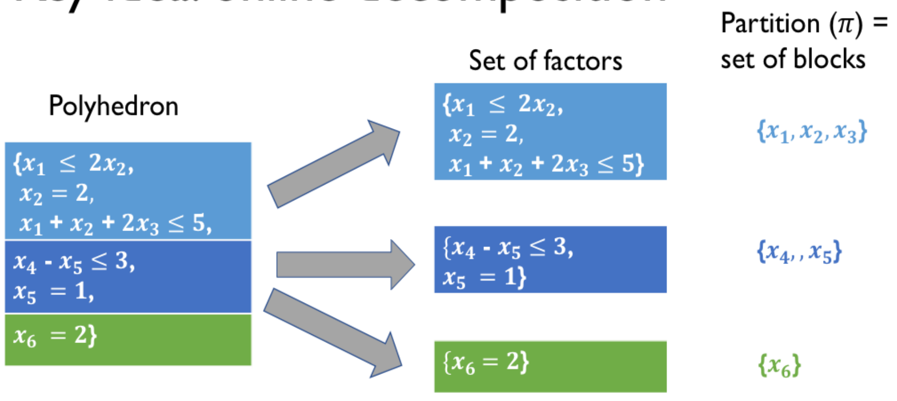
Inclusion (⊑)	$O(ngm)$	$O(\sum_{i=1}^{r} n_i m_i g_i)$
Assignment	$O(ng)$	$O(n_{max} g_{max})$
Conditional	$O(n)$	$O(n_{max})$

ELINA

http://elina.ethz.ch

Conclusion

Key idea: online decomposition



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ELINA

http://elina.ethz.ch

Driver	NewPolka	PPL	ELINA
➢ 500 var ➢ 39K LOC	OOM (> 12 GB)	OOM (> 12 GB)	4 sec 0.9 GB
➢ 650 var ➢ 25K LOC	TO (> 4 hr)	TO (> 4 hr)	2 sec 0.4 GB